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Stability properties of the collective stationary motion of self-propelling particles with conservative kinematic constraints

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Abstract
In our previous papers we proposed a continuum model for the dynamics of the systems of self-propelling particles with conservative kinematic constraints on the velocities. We have determined a class of stationary solutions of this hydrodynamic model and have shown that two types of stationary flow, linear and axially symmetric (vortical) flow, are possible. In this paper we consider the stability properties of these stationary flows. We show, using a linear stability analysis, that the linear solutions are neutrally stable with respect to the imposed velocity and density perturbations. A similar analysis of the stability of the vortical solution is found to be not conclusive.

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1. Introduction

The dynamics of the systems of self-propelling particles (SPP) is of a great interest for physicists as well as for biologists because of the complex and fascinating phenomenon of the emergence of the ordered motion. In nature these systems are represented by flocks of birds, schools of fishes, groups of bacteria, etc [1, 3]. From the physical point of view many aspects of the observed non-equilibrium phase transition from disordered to ordered motion are to a large extent still an open problem.

The first numerical model simulating the behaviour of the SPP was proposed by Vicsek *et al* [4]. For shortness we call it the Czirók–Vicsek automaton or algorithm (CVA). The model is based on a kinematic rule imposed on the orientations of the velocities of the self-propelling particles. At a low-noise amplitude and a high density it was shown that the system undergoes the transition from the disordered state to coherent motion. There is strong support that the transition is of continuous character which assumes that one may use the approach similar to that for equilibrium continuous order–disorder phase transitions. In particular, the powerful technique of renormgroup analysis allows us to get the critical exponents for the velocity correlation functions [5, 6]. It is natural that the investigation of the continuous models in different aspects for self-propelling systems at least allows us to avoid the difficulties in distinguishing the physical factors from the artificial numerical effect [7].

The dynamics of Vicsek's model was also investigated in the framework of graph theory. In [8], the spontaneous emergence of ordered motion has been studied in terms of so-called control laws which govern the dynamics of the particles. Generalizations of the control laws were considered in [9, 10]. In particular, in [10] it was shown that the organized motion of SPP with the control laws depending on the relative orientations of the velocities and relative spacing can be of two types only: parallel and circular motion. The stability properties of these discrete updating rules (including Vicsek's model) and the dynamics they describe were considered using Lyapunov theory in [8, 9, 11].

In our first paper [12] we constructed a hydrodynamic model for the system of self-propelling particles with conservative kinematic constraints, which can be considered as a continuum analogue of the discrete dynamic automaton proposed by Vicsek *et al*.

Based on the conservation of the kinetic energy and the number of particles our model is represented by the following equations:

$$\frac{\mathrm{d}\mathbf{v}(\mathbf{r},t)}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{r},t) \times \mathbf{v}(\mathbf{r},t), \tag{1}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)) = 0, \tag{2}$$

where $\mathbf{v}(\mathbf{r},t)$ and $n(\mathbf{r},t)$ are the velocity and the density fields respectively, and $\boldsymbol{\omega}(\mathbf{r},t)$ is an angular velocity field which takes into account the non-potential character of the interactions between the particles. We modelled this field as follows:

$$\boldsymbol{\omega}(\mathbf{r},t) = \int K_1(\mathbf{r}-\mathbf{r}')n(\mathbf{r}',t)\,\mathrm{rot}\,\mathbf{v}(\mathbf{r}',t)\,\mathrm{d}\mathbf{r}' + \int K_2(\mathbf{r}-\mathbf{r}')\nabla n(\mathbf{r}',t) \times \mathbf{v}(\mathbf{r}',t)\,\mathrm{d}\mathbf{r}', \tag{3}$$

where $K_{1,2}(\mathbf{r}-\mathbf{r}')$ are the averaging kernels. In particular, we considered a simple case of averaging kernels:

$$K_i(\mathbf{r}-\mathbf{r}') = s_i\delta(\mathbf{r}-\mathbf{r}'), \qquad \text{where} \quad i = 1 \text{ or } 2. \tag{4}$$

We call this the local hydrodynamic model (LHM). In such a case one may consider such a continuum model as the particular case of the general hydrodynamical model considered in [5] obeying the conservation rules of the CVA. The viscous Navier–Stokes term is absent because of the dissipative-free character of the dynamics. In fact for the CVA the energy of the chaotic motion at the low-noise level still can be transformed into the ordered motion. While the viscosity for the ordinary fluid transmits the energy of the ordered motion into the heat. In this case equation (3) reduces to

$$\boldsymbol{\omega}(\mathbf{r},t) = s_1 n(\mathbf{r},t)\,\mathrm{rot}\,\mathbf{v}(\mathbf{r},t) + s_2\nabla n(\mathbf{r},t) \times \mathbf{v}(\mathbf{r},t), \tag{5}$$

where

$$s_i = \int K_i(\mathbf{r})\,\mathrm{d}\mathbf{r}. \tag{6}$$

In our second paper [13] we have shown that the only regimes of the stationary planar motion in our model are either of translational or axial symmetry. In this respect our continuum model gives results similar with those obtained in the discrete model of Vicsek [4, 14].

In this paper we investigate the stability of the obtained regimes of motion with respect to small perturbations. In the following section we consider the stability of the planar stationary linear flow with respect to the velocity perturbation directed along the stationary flow and perpendicular to the flow. We show that in both cases the evolution of the perturbations has an oscillatory behaviour, which means that they neither grow nor decay with time. This can be interpreted as neutral stability [15] of the corresponding stationary flow. Also the external pressure term $-\nabla p/n$ can be included into equation (1) in order to account for potential external forces. In such a case with $s_2 = 0$ there exists the special case of the incompressible flows, $n = \text{const}$, when the equations of motion (1), (2) with (5) coincide with those for potential flow of ideal fluids. As is known [15], such motion in 2D geometry is stable under rather weak restrictions on the initial velocity field profile in the Lyapunov sense.

In the third section we consider the stability of the planar stationary axially symmetric (vortical) motion of SPP with constant velocity and the density. We find that in this case the linear analysis does not lead to a conclusive answer about the stability of the solution.

2. Stability of planar stationary linear flow in the local hydrodynamic model

2.1. Stability with respect to a velocity perturbation along the flow

In this section we consider the stability properties of planar stationary linear flow for the local hydrodynamic model with $s_2 = 0$, which we further call local the hydrodynamic model 1 (LHM1). At the end of the section we will shortly discuss how these results extend to the local hydrodynamic models with $s_1 = 0$ and $s_1 = s_2$. For LHM1, the stationary linear flow is given by

$$\mathbf{v}_0(\mathbf{r}) = v_0\mathbf{e}_x \qquad \text{and} \qquad n_0(\mathbf{r}) = n_0, \tag{7}$$

where v_0 and n_0 are constants.

We consider velocity and the density perturbations of the following form:

$$\mathbf{v}_1(\mathbf{r}, t) = v_0 A_\parallel \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \, \mathrm{e}^{\alpha_\parallel t} \, \mathbf{e}_x \qquad \text{and} \qquad n_1(\mathbf{r}, t) = n_0 B_\parallel \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \, \mathrm{e}^{\alpha_\parallel t}. \tag{8}$$

The velocity perturbation chosen is directed along the stationary linear flow. Here $A_\parallel$, $B_\parallel$ are constants, $\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y$ is the wave vector and $\alpha_\parallel$ is an exponent, which determines the time evolution of the perturbation.

Substituting the solution $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0 + \mathbf{v}_1(\mathbf{r}, t)$, $n(\mathbf{r}, t) = n_0 + n_1(\mathbf{r}, t)$ into equations (1) and (2) we obtain the linearized system of equations

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1 = s_1 n_0 (\mathrm{rot}\,\mathbf{v}_1) \times \mathbf{v}_0, \tag{9}$$

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0\mathbf{v}_1) + \nabla \cdot (n_1\mathbf{v}_0) = 0. \tag{10}$$

For perturbation (8) this system reduces to

$$\frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial x} = 0, \tag{11}$$

$$\frac{\partial v_1}{\partial y} = 0, \tag{12}$$

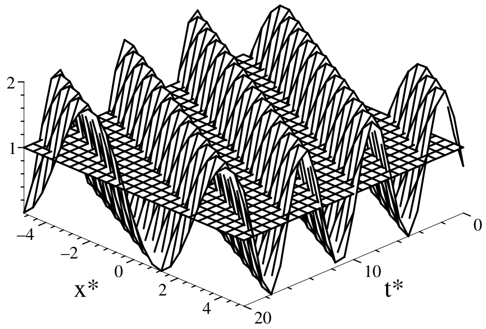

Figure 1. The total density field $n(\mathbf{r}, t)/n_0$ and the stationary solution $n/n_0 = 1$ as a function of $x^* = k_x x$ and $t^* = k_x v_0 t$ for $k_y = 0$.

$$\frac{\partial n_1}{\partial t} + v_0 \frac{\partial n_1}{\partial x} + n_0 \frac{\partial v_1}{\partial x} = 0. \tag{13}$$

Using equation (8) one may obtain the relation between $\alpha_\parallel$ and the wave number. From equation (11) it follows that

$$\alpha_\parallel = -\mathrm{i} k_x v_0, \tag{14}$$

whereas from the linearized continuity equation (13) we have

$$\alpha_\parallel = -\mathrm{i} k_x v_0 \frac{(A_\parallel + B_\parallel)}{B_\parallel}.$$

Both the equalities are satisfied only in the case when $A_\parallel = 0$.

Thus, in the linear stability analysis with respect to small deviations of the velocity and density fields, we obtain the following perturbed solution:

$$\mathbf{v} = v_0\, \mathbf{e}_x, \qquad n = n_0 [1 + B_\parallel\, \mathrm{e}^{\mathrm{i} k_y y}\, \mathrm{e}^{\mathrm{i} k_x (x - v_0 t)}]. \tag{15}$$

Taking the real part of the density perturbation we have

$$\mathbf{v} = v_0\, \mathbf{e}_x, \qquad n = n_0 [1 + B_\parallel \cos(\mathbf{k} \cdot \mathbf{r} - k_x v_0 t)]. \tag{16}$$

The corresponding density field is shown in figure 1.

This flow equation (16) should satisfy the linearized system of the constraints (conservation of the kinetic energy and the number of particles) which are imposed on any solution of our model. This implies that the following conditions must be fulfilled:

$$\int n_1(\mathbf{r}, t)\, \mathrm{d}\mathbf{r} = 0, \tag{17}$$

$$\int \left[2 n_0 (\mathbf{v}_0 \cdot \mathbf{v}_1(\mathbf{r}, t)) + n_1(\mathbf{r}, t) v_0^2\right] \mathrm{d}\mathbf{r} = 0. \tag{18}$$

Since $\mathbf{v}_1(\mathbf{r}, t) = \mathbf{0}$ both conditions reduce to

$$\int n_1(\mathbf{r}, t)\, \mathrm{d}\mathbf{r} = n_0 B_\parallel \int \mathrm{e}^{\mathrm{i} k_y y}\, \mathrm{d}y \int \mathrm{e}^{\mathrm{i} k_x (x - v_0 t)}\, \mathrm{d}x = 0. \tag{19}$$

If one integrates equation (19) over the period of the integrand, one may see that this condition is fulfilled.

The obtained perturbed flow is an oscillatory field (perturbation oscillates with a frequency $\alpha_\parallel$ as $t \to \infty$) which means that the corresponding stationary solution is neither stable nor unstable within the first-order perturbation theory. In other words, we may conclude that in our local hydrodynamic model the stationary linear flow is neutrally stable with respect to a small density field perturbations.

The stability analysis of the other possible hydrodynamic models with $s_1 = 0$ or $s_1 = s_2$ gives qualitatively similar result.

2.2. Stability with respect to a velocity perturbation perpendicular to the flow

In this section we investigate the stability properties of the stationary linear flow in the LHM1, equation (7), with respect to a velocity perturbation normal to the stationary flow. We consider only a velocity perturbation, which we take in the form of a plane wave:

$$\mathbf{v}_1 = v_0 A_\perp \,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{e}^{\alpha_\perp t}\,\mathbf{e}_y, \qquad n_1 = 0, \tag{20}$$

where $A_\perp$ is a constant, $\mathbf{k}$ is a wave vector and the exponent $\alpha_\perp$ describes the time evolution of the perturbation.

Substituting the perturbation in the linearized equations (9) and (10) it follows that

$$\frac{\partial v_1}{\partial t} + v_0(1 - s_1 n_0)\frac{\partial v_1}{\partial x} = 0, \tag{21}$$

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{and} \qquad k_y = 0, \tag{22}$$

which imply that

$$\alpha_\perp = \mathrm{i}k_x v_0(s_1 n_0 - 1). \tag{23}$$

Thus the time evolution of the perturbed velocity field is determined by the purely imaginary exponent in equation (23):

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1(x, t) = v_0[\mathbf{e}_x + A_\perp \,\mathrm{e}^{\mathrm{i}k_x(x+\mathbb{V}t)}\,\mathbf{e}_y], \qquad n = n_0, \tag{24}$$

where the 'phase speed' is given by

$$\mathbb{V} = v_0(s_1 n_0 - 1).$$

Taking the real part of the velocity perturbation we obtain as the final result

$$\mathbf{v} = v_0[\mathbf{e}_x + A_\perp \cos[k_x(x + \mathbb{V}t)]\,\mathbf{e}_y], \qquad n = n_0. \tag{25}$$

The corresponding velocity profile is shown in figure 2.

Since the velocity perturbation was taken to be normal to the unperturbed field and $n_1 = 0$, both of the constraints of the constancy of the kinetic energy and the number of particles, equations (17) and (18), are satisfied.

As one may see the time-dependent part of the velocity perturbation is a finite oscillatory function which means that the corresponding stationary solution is neutrally stable.

As in the previous section the stability analysis of the other possible hydrodynamic models with $s_1 = 0$ or $s_1 = s_2$ gives qualitatively similar result.

3. Stability of stationary vortical flow with constant velocity and density in the local hydrodynamic model

As we have shown in our previous article [13] there are two classes of the stationary flows in the LHM, linear and axially symmetric or vortical flow.

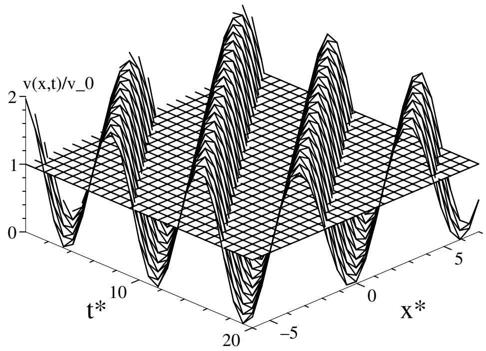

Figure 2. The total velocity field $v(x,t)/v_0$ and the stationary velocity field $v_0/v_0 = 1$ as a function of $x^* = k_x x$ and $t^* = k_x \mathbb{V} t$.

The stationary vortical solution of the LHM1 ($s_2 = 0$) is given by $\mathbf{v}_0(\mathbf{r}) = v_\varphi(r)\,\mathbf{e}_\varphi$, $n_0(\mathbf{r}) = n_0(r)$, [12], where

$$v_\varphi(r) = \frac{C_{\rm st}}{2\pi r}\exp\left[s_1\int_{r_0}^{r}\frac{{\rm d}r'}{r' n_0(r')}\right]. \qquad (26)$$

Here r_0 is a cut-off radius of the vortex core and the constant $C_{\rm st}$ is determined by the circulation of the core

$$\oint_{r=r_0}\mathbf{v}\,{\rm d}\mathbf{l} = C_{\rm st}. \qquad (27)$$

We consider small perturbations $\mathbf{v}_1(r,\varphi,t)$ of the velocity field and $n_1(r,\varphi,t)$ of the density field. The linearized system in the LHM1 is then given by

$$\frac{\partial\mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 = s_1 n_0[(\mathrm{rot}\,\mathbf{v}_1)\times\mathbf{v}_0 + (\mathrm{rot}\,\mathbf{v}_0)\times\mathbf{v}_1] + s_1 n_1(\mathrm{rot}\,\mathbf{v}_0)\times\mathbf{v}_0, \qquad (28)$$

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(n_0\mathbf{v}_1) + \nabla\cdot(n_1\mathbf{v}_0) = 0. \qquad (29)$$

In this section we consider the stability of a particular class of stationary vortical flow for which the density is constant and given by $n_0 = 1/s_1$. Substitution in equation (26) results in a constant velocity field $\mathbf{v}_0 = v_\varphi\,\mathbf{e}_\varphi = (C_{\rm st}/2\pi r_0)\,\mathbf{e}_\varphi \equiv C\,\mathbf{e}_\varphi$. We write the small perturbation in the general form

$$\mathbf{v}_1 = a(r,\varphi,t)\,\mathbf{e}_r + b(r,\varphi,t)\,\mathbf{e}_\varphi \qquad \text{and} \qquad n_1 = n_0 c_1(r,\varphi,t). \qquad (30)$$

For the projections of the velocity field $\mathbf{v} = \mathbf{v}_0(r) + \mathbf{v}_1(r,\varphi,t)$ together with the continuity equation for the density field $n = n_0 + n_1(r,\varphi,t)$ we have

$$\frac{\partial a}{\partial t} - 2\frac{b v_\varphi}{r} + \frac{v_\varphi}{r}\frac{\partial a}{\partial\varphi} = -\frac{v_\varphi}{r}\left[\frac{\partial r b}{\partial r} - \frac{\partial a}{\partial\varphi}\right] - \frac{b v_\varphi}{r} - c_1\frac{v_\varphi^2}{r}, \qquad (31)$$

$$\frac{\partial b}{\partial t} + \frac{v_\varphi}{r}\frac{\partial b}{\partial\varphi} = 0, \qquad (32)$$

$$\frac{\partial c_1}{\partial t} + \frac{1}{r}\left[\frac{\partial r a}{\partial r} + \frac{\partial b}{\partial\varphi}\right] + \frac{v_\varphi}{r}\frac{\partial c_1}{\partial\varphi} = 0. \qquad (33)$$

In order to simplify the problem we restrict our discussion to the case with the radial component of the velocity perturbation being constant, i.e. $a(r, \varphi, t) = \text{const}$.

Then one can transform equations (31)–(33) into

$$\frac{\partial b}{\partial t} + \frac{v_\varphi}{r}\frac{\partial b}{\partial \varphi} = 0, \tag{34}$$

$$\frac{\partial b}{\partial r} = -\frac{c_1 v_\varphi}{r}, \tag{35}$$

$$\frac{\partial c_1}{\partial t} + \frac{1}{r}\left(a + \frac{\partial b}{\partial \varphi}\right) + \frac{v_\varphi}{r}\frac{\partial c_1}{\partial \varphi} = 0. \tag{36}$$

The velocity perturbation must be a periodic functions of the angle φ and can therefore be written as

$$b(r, \varphi, t) = v_\varphi B(r)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{e}^{\beta t}, \tag{37}$$

where $B(r)$ is a function of r, m is an integer and β is a constant factor, which describes the time evolution of the perturbation, equation (30). Substituting this into equation (34) one obtains

$$\beta = -\mathrm{i}m\frac{v_\varphi}{r} \tag{38}$$

and consequently

$$b(r, \varphi, t) = v_\varphi B(r) \exp\left[\mathrm{i}m\left(\varphi - \frac{v_\varphi}{r}t\right)\right]. \tag{39}$$

From equation (35) it follows that

$$c_1(r, \varphi, t) = -r\left(\frac{\partial B(r)}{\partial r} + \mathrm{i}m\frac{v_\varphi B(r)}{r^2}t\right)\exp\left[\mathrm{i}m\left(\varphi - \frac{v_\varphi}{r}t\right)\right]. \tag{40}$$

Substituting this into equation (36) we obtain that $a(r, \varphi, t) = 0$.

Solutions (39) and (40) satisfy the linearized system of constraints, equations (17) and (18), as one can see by angular integration.

Thus, we see that the time evolution of the perturbation equation (30) is determined by the purely imaginary exponent equation (38).

Taking the real part in equations (39) and (40) we obtain

$$b(r, \varphi, t) = v_\varphi B(r)\cos\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right], \tag{41}$$

$$n_1(r, \varphi, t) = n_0\left\{\frac{m v_\varphi B(r)}{r}t\sin\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right] - r\frac{\partial B(r)}{\partial r}\cos\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right]\right\}. \tag{42}$$

As a result the whole solution for the velocity and the density profiles has the following form:

$$\mathbf{v}(r, \varphi, t) = v_\varphi\left\{1 + B(r)\cos\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right]\right\}\mathbf{e}_\varphi, \tag{43}$$

$$n(r, \varphi, t) = n_0\left\{1 + \frac{m v_\varphi B(r)}{r}t\sin\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right] - r\frac{\partial B(r)}{\partial r}\cos\left[m\left(\varphi - \frac{v_\varphi}{r}t\right)\right]\right\}. \tag{44}$$

The velocity field is shown in figure 3 for $m = 1$ and $r = 5$ m.

Together with the oscillatory contributions we now also have the contribution proportional to t times an oscillating function. This does not necessarily mean that the stationary vortical

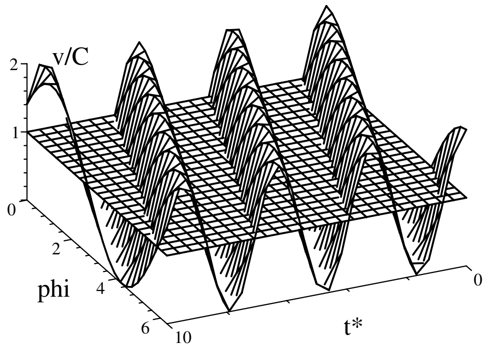

Figure 3. The total velocity field $v(r, \varphi, t)/v_\varphi$ and $v_\varphi(r)/v_\varphi = 1$ as a function of φ and $t^* = v_\varphi t/r$ for $m = 1$ and $r = 5$ m.

flow is unstable. The linear analysis does not give the definitive answer regarding the stability of the stationary flow, and further investigation of higher order terms is required. Note that such a situation is typical for Hamiltonian systems which are conservative by definition and therefore do not display an asymptotic type of stability [15]. Though the system under consideration is not Hamiltonian, one may suppose that the reason for the neutral stability is the dissipative-free character of the dynamics.

4. Conclusions

In this paper, we considered the stability properties of the planar stationary flows of the local hydrodynamic model constructed in our first paper for a system of self-propelling particles [12]. These flows are the linear flow and the axially symmetric flow. Our analysis shows for linear flow, using linear perturbation theory, that the time evolution of the imposed velocity and density perturbations are oscillatory. It follows that the linear flows are neutrally stable. For axially symmetric (vortical) flow linear perturbation theory does not lead to a conclusive result. A definitive answer about the nature of the stability can only be given by considering also higher order terms in the perturbation expansion. Such an analysis is beyond the scope of the present paper.

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